

# Playing The Hypothesis Testing Minority Game In The Maximal Reduced Strategy Space

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Hypothesis Testing Minority Game (HMG) is a variant of the standard Minority Game (MG) that models the inertial behavior of agents in the market. In the earlier study of our group, we find that agents cooperate better in HMG than in the standard MG when strategies are picked from the full strategy space. Here we continue to study the behavior of HMG when strategies are chosen from the maximal reduced strategy space. Surprisingly, we find that, unlike the standard MG, the level of cooperation in HMG depends strongly on the strategy space used. In addition, a novel intermittency dynamics is also observed in the minority choice time series in a certain parameter range in which the orderly phases are characterized by a variety of periodic dynamics. Remarkably, all these findings can be explained by the crowd-anticrowd theory.

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## I. INTRODUCTION

Studying economic systems by agents-based models have attracted the attention among physicists in recent years [1, 2, 3]. One of the most famous agents-based model in this regard is the Minority Game (MG) [3, 4]. MG does not only capture the fact that all people in the market think inductively and selfishly [5], its complexity also satisfy the definition of a complex system in the strictest sense [6]. In spite of its simple governing rules, agents in this model self-organize giving rise to an unexpected global cooperative phenomena.

Using the standard MG as blue print, various modifications to the rules of the standard MG have been proposed [7, 8, 9, 10] to understand different aspects and phenomena in realistic economic systems. In particular, Man and Chau introduced the Hypothesis Testing Minority Game (HMG) to model the inertial behavior of agents [11]. They found that the presence of inertial agents improve global cooperation leading to a decrease of the variance per agent over the entire parameter space provided that the strategies of each agent are chosen from the so-called full strategy space (FSS) [11, 12].

In this paper, we move on to study the agent cooperation and the dynamics of HMG in case the strategies are picked from the so-called maximal reduced strategy space (MRSS) [13]. We begin by briefly reviewing the rules of HMG and stating the parameters used in our numerical simulations in Sec. II. Then we report our simulation results in Sec. III. To our surprise, the behavior of HMG depends strongly on the strategy spaces used. Specifically, agents generally cooperate better when strategies are picked from the FSS rather than the MRSS provided that they are sufficiently reluctant to change their strategies. In contrast, the standard MG is so robust that its dynamics and cooperative behavior are essentially the same irrespective of whether the FSS or the MRSS is used. Furthermore, we find that in HMG the minority

choice time series exhibits intermittency in which the orderly phases show periodic dynamics with period up to  $2(2^M - 1)$  whenever the memory size of the strategies  $M$  is greater than 1 in a certain parameter range when strategies are picked from the MRSS. This novel intermittent phenomenon does not show up in HMG provided that strategies are picked from the FSS as well as in the standard MG. We explain how these differences originate from the choice of the strategy space by a semi-analytical approach known as the crowd-anticrowd theory [14, 15] in Sec. IV. In fact, the major reason responsible for these differences is that it is a lot easier for an agent to keep on using one's currently adopted strategy when the strategy pool is the MRSS than rather than FSS in certain parameter regime. Finally, we summarize our findings in Sec. V. Our findings show that extra care is needed to study variants of MG as their behavior may depend sensitively on the strategy space employed. Nonetheless, the ability to explain the behavior of HMG using the crowd-anticrowd theory suggests that this theory may still be useful to explain the dynamics of variants of the standard MG provided that one carries out the analysis carefully.

## II. HYPOTHESIS TESTING MINORITY GAME

Recall that in the standard MG, agents act according to the predictions of their best performing strategies. In other words, agents in the standard MG do not hesitate to stop using their current strategies once the performance indicator, known as virtual score, shows that the strategies are not the best. In contrast, the HMG incorporated the inertial behavior of agents by allowing them to stick to their currently using strategies until their performances are too poor to be acceptable. More precisely, a fixed real number  $I_k$  between 0.5 and 1.0 is assigned once and for all to each agent  $k$  in HMG to represent their reluctance to switch strategies. Using the value of

$I_k$  as an indicator of the confidence level, agent  $k$  tests the hypothesis that his currently using strategy is his best strategy at hand at each turn. Furthermore, he switches to another strategy and resets the virtual scores of all his strategies to 0 if the null hypothesis is rejected. Apart from these differences, the governing rules of HMG are identical to those of the standard MG.

We state the rules of HMG below for reader's convenience.

### A. Rules of the game

1. HMG is a repeated game of a fixed population of  $N$  agents. A number  $I_k \in [0.5, 1)$  is assigned to agent  $k$  once and for all to represent his inertia.
2. At each turn  $\tau$ , every agent has to make a choice between one of the two sides (namely side 0 and side 1) based on the strategies to be described in rule 4. Those agents in the side with the least number of agents (known as the minority side) win in that turn. And in case of a tie, the winning side is randomly selected.
3. The only piece of global information reveals to the agents at time  $\tau$  is the winning sides in the last  $M$  turns known as the history  $\vec{\mu}(\tau)$ .
4. Before the game commences, each agent  $k$  is assigned once and for all  $S$  randomly picked strategies  $S_{k,i}$  for  $i = 0, 1, \dots, S-1$ . Each strategy  $S_{k,i}$  is a function map from the set of all possible histories to the set  $\{0, 1\}$  and its virtual score  $\Omega_{k,i}$  is set to 0 initially. Without loss of generality, strategy  $S_{k,0}$  is assumed to be the currently using strategy of agent  $k$  at the beginning of game.
5. Agent  $k$  will switch his current strategy from  $S_{k,0}$  to  $S_{k,j}$  if and only if the maximum virtual score difference  $\Delta\Omega_k$  drops below the threshold  $x_k\sqrt{2\tau_k}$ , that is,

$$\begin{aligned} \Delta\Omega_k &\equiv \max_i \{\Omega_{k,0} - \Omega_{k,i}\} = \Omega_{k,0} - \Omega_{k,j} \\ &\leq x_k \sqrt{2\tau_k} \end{aligned} \quad (1)$$

where  $x_k$  is defined by

$$\frac{1}{\sqrt{2\pi}} \int_{x_k}^{+\infty} e^{-x^2/2} dx = I_k \quad (2)$$

and  $\tau_k$  is the number of turns elapsed since agent  $k$ 's last switch of strategy. In case agent  $k$  switches his strategy, he exchanges the labels 0 and  $j$  so that his currently using strategy is always labeled as  $S_{k,0}$ . In addition, the virtual scores  $\Omega_{k,i}$  are reset to 0 for all  $i$  and  $\tau_k$  is reset to 1.

6. Agent  $k$  uses his current strategy to guess the minority choice of the current turn. Moreover, the virtual score of strategy  $S_{k,i}$  is increased (decreased) by 1 if it predicts the minority side of that turn correctly (incorrectly).

### B. Parameters used in our simulation

We select the following parameters in our simulations:

1.  $N$  is odd;
2.  $S = 2$ ;
3. all values of  $I_k$  are chosen to be the same independent of the agent label  $k$  (and we write this common  $I_k$  as  $I$  for simplicity); and
4.  $S_{k,j}$  are picked from the so-called MRSS [13]. (That is,  $S_{k,j}$  can be written in the form

$$s_\tau = \eta_0 + \sum_{i=1}^M \eta_i \mu_{\tau-i}, \quad (3)$$

where  $s_\tau$  is the prediction of the minority side in the  $\tau$ th turn,  $\eta_0, \eta_1, \dots, \eta_M \in \{0, 1\}$ ,  $\mu_i$  is the minority side in the  $i$ th turn and the arithmetic is performed in the finite field of two elements  $GF(2)$  [8, 13, 16]. In other words, a strategy in the MRSS is characterized by  $(\eta_0, \eta_1, \dots, \eta_M)$ .

With the exception of point 4, the parameters used in this study are identical to those used in our earlier study of HMG reported in Refs. [11, 12]. In contrast, strategies in Refs. [11, 12] are picked from the FSS, namely, the set of all possible strategies. And a strategy in the FSS may not be expressed as a linear function of  $\mu_i$ 's. From now on, we use the symbols  $\text{HMG}^{\text{FSS}}$  and  $\text{HMG}^{\text{MRSS}}$  to denote the HMG in which strategies are picked from the FSS and the MRSS, respectively.

## III. OUR NUMERICAL SIMULATION RESULTS

### A. Focus of our study

We are interested in both the cooperative behaviors and the dynamics of the game. Recall that MG and HMG are non-positive sum games in the sense that the number of winning agents is less than or equal to the number of losing agents in each turn. And we say that the agents (or the system) cooperate better if the average number of winning agents per turn is high. Our numerical simulations show that agents self-organize in such a way that there is no bias in picking the minority side when averaged over the agents and the number of turns, so we follow the usual practice to study agent co-operation by means of the  $\alpha \equiv 2^{M+1}/NS$  against  $\sigma^2/N$

graph where  $\sigma^2$  is the variance of the number of agents choosing side 0 [13]. The lower the value of  $\sigma^2/N$ , the better the agent cooperation.

As for the dynamics of HMG, our investigation focuses on the analysis of the periodicity of the minority choice time series through the auto-correlation function. And the auto-correlation function can be conveniently studied by means of a time lag  $t$  against auto-correlation  $C_0$  graph. In order to make sure that the dynamics is genuine and long lasting, we only consider the time series after the system has equilibrated. We also perform simulations using different values of  $N$  and initial quenched disorders to make sure that the dynamics we are going to report below are generic.

Actually, the dynamics depends on the following three factors:

1. number of agents  $N$ ;
2. history size  $M$ ; and
3. the initial quenched disorder as reflected by the value  $I$  and the strategies  $S_{k,i}$  assigned to the agents.

Our choice of parameters for the HMG<sup>MRSS</sup> reported in Sec. II B makes the dynamics of the game deterministic and hence enabling us to study the periodic dynamics of the minority choice time series easily. In contrast, when played using other choices of  $N$  and  $S$ , the non-deterministic nature of this game weakens the periodic dynamics in minority choice time series, making both the numerical and analytical studies more troublesome.

Unlike the standard MG, we find that both the cooperative behavior and the dynamics of HMG depend strongly on the strategy space chosen. We shall elaborate more on this point in the coming two subsections.

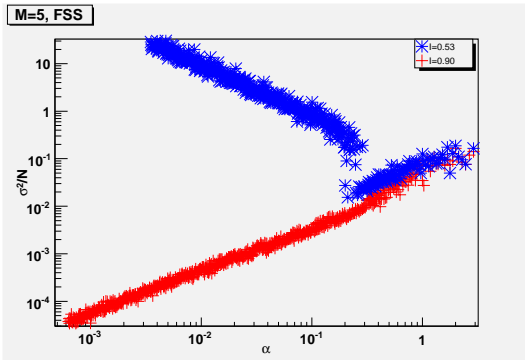


FIG. 1: The  $\alpha$  against  $\sigma^2/N$  graph for HMG<sup>FSS</sup>. The value of  $\alpha$  is varied by fixing  $M$  and changing  $N$ .

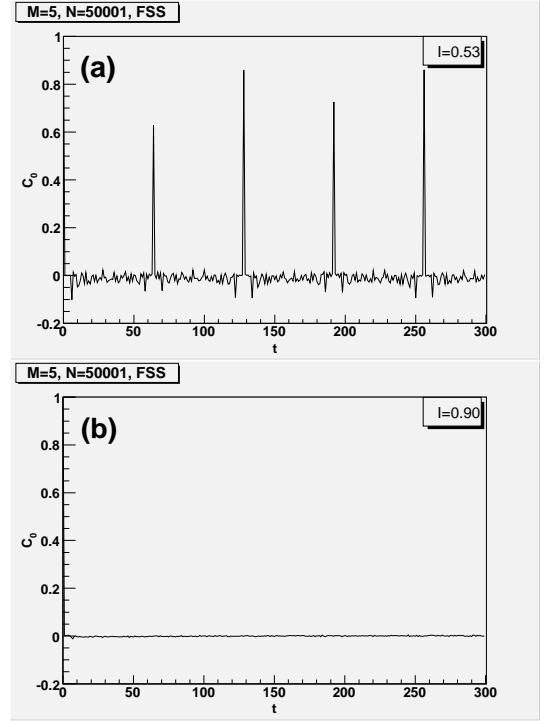


FIG. 2: The auto-correlation for HMG<sup>FSS</sup> with  $\alpha = 6.4 \times 10^{-4} \ll \alpha_c^{\text{MG}}$  when (a)  $I = 0.53$  and (b)  $I = 0.90$ .

## B. Reviewing the simulation results of the HMG<sup>FSS</sup>

Recall from the earlier study of our group in Refs. [11, 12] that when  $I$  is chosen to be less than  $I^{c1}$ , where

$$I^{c1} = \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{2/2^{M+1}}}^{+\infty} e^{-x^2/2} dx, \quad (4)$$

the inertia of agents is not strong enough to make the dynamics of the HMG<sup>FSS</sup> to deviate significantly from that of the standard MG. Thus, the  $\sigma^2/N$  is about the same as that of the standard MG. Moreover, the well-known period  $2^{M+1}$  dynamics in the minority choice time series that appears in the standard MG when  $\alpha$  is less than  $\alpha_c^{\text{MG}} \approx 0.3$ , the critical value for the standard MG, is also present here [11, 12]. In contrast, when  $I > I^{c1}$ , the inertia of agents becomes strong enough to significantly reduce the herd effect amongst agents resulting in a much lower  $\sigma^2/N$  (and hence indicating that agents cooperate better). Besides, the period  $2^{M+1}$  dynamics is no longer present when  $\alpha < \alpha_c^{\text{MG}}$  [11, 12]. These earlier findings are summarized in Figs. 1 and 2.

## C. Simulation results of the HMG<sup>MRSS</sup>

Contrary to our expectation, we find that the behavior of HMG<sup>MRSS</sup> is significantly different from that of

HMG<sup>FSS</sup> when  $I \gtrsim I^{c1}$  and  $\alpha \lesssim 1$ . The details of our findings are listed below.

- $I \lesssim I^{c1}$ : By comparing Fig. 1 with Fig. 3a and Fig. 2 with Figs. 4 and 5, we know that the behavior of HMG<sup>MRSS</sup> in this regime is similar to that of HMG<sup>FSS</sup>. That is, they have about the same level of agent cooperation. In addition, the standard MG [17, 18, 19], HMG<sup>FSS</sup> [11] and HMG<sup>MRSS</sup> all exhibit the same period  $2^{M+1}$  dynamics in the minority choice time series whenever  $\alpha \lesssim \alpha_c^{\text{MG}}$  and show no periodic dynamics for  $\alpha \gtrsim \alpha_c^{\text{MG}}$ .
- $I \gtrsim I^{c1}$  and  $\alpha \geq 1 (\gg \alpha_c^{\text{MG}})$ : By comparing Fig. 1 with Fig. 3, we find that the values of  $\sigma^2/N$  are about the same for both HMG<sup>FSS</sup> and HMG<sup>MRSS</sup> in this regime. Moreover, the standard MG [13], HMG<sup>FSS</sup> [11] and HMG<sup>MRSS</sup> all show no periodic dynamics.
- $I \gtrsim I^{c1}$  and  $\alpha \lesssim 0.1 (\ll \alpha_c^{\text{MG}})$ : Figs. 1 and 3 show that for the same value of  $\alpha$  in this regime, the cooperation amongst agents for the standard MG is the worst, for HMG<sup>MRSS</sup> is in the middle and for HMG<sup>FSS</sup> is the best. One interesting feature for HMG<sup>MRSS</sup> is that, unlike HMG<sup>FSS</sup>, the value of  $\sigma^2/N$  increases as  $\alpha$  decreases in this regime indicating that agents cooperate less and less as the number of agents  $N$  increases (and with  $M$  and  $S$  held fixed). As for the dynamics, Fig. 6 depicts that the system exhibits no obvious periodic dynamics.

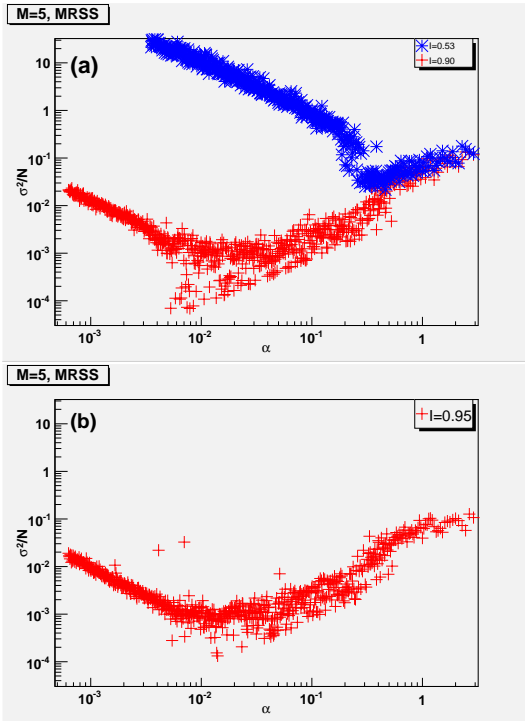


FIG. 3: Plots of  $\alpha$  against  $\sigma^2/N$  for HMG<sup>MRSS</sup> with (a)  $I = 0.53, 0.90$  and (b)  $I = 0.95$ .

Nonetheless, its minority choice time series conditioned on an arbitrary but fixed history exhibits a very weak period two dynamics.

- $I \gtrsim I^{c1}$  and  $0.1 \lesssim \alpha \approx \alpha_c^{\text{MG}} < 1$ : In this regime, we find that the value of  $\sigma^2/N$  obtained after equilibration depends on the initial quenched disorder of the system indicating the presence of a phase transition point. (See Fig. 3.) Actually, the values of  $\sigma^2/N$  obtained in many runs are rather close to the theoretical minimum of  $1/4N$  (which is attained when there are exactly  $(N-1)/2$  winning agents in each turn) implying that agents cooperate almost perfectly.

The dynamics of the minority choice time series in this regime is rather complex. Actually, no obvious periodic dynamics is observed for those initial quenched disorder that ends up with values of  $\sigma^2/N$  about the same as those for  $I < I^{c1}$ . (See Fig. 7a.) In contrast, those ending up with a much smaller  $\sigma^2/N$  show intermittency. (See Fig. 7b.) That is to say, when  $\sigma^2/N$  is small, the time series exhibits periodic dynamics for some time and then the periodicity either suddenly disappears or the period of the dynamics changes. Also the brief episode of aperiodicity terminates with the commencement of a new periodic dynamics (with possibly a new period).

Interestingly, the period of the orderly phase for this intermittency depends on the value of  $I$ . In case  $I^{c1} \lesssim I < I^{c2}$ , where

$$I^{c2} \equiv \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{2}}^{+\infty} e^{-x^2/2} dx \approx 0.92135, \quad (5)$$

the periods are less than or equal to  $2(2^M - 1)$  whenever  $M \geq 2$ . More importantly, these periods are in the form  $2^j \text{L.C.M.}(p_1, p_2, \dots, p_M)$  where  $j \geq 1$  is an integer, L.C.M. denotes the least common multiple of the  $M$  arguments and  $p_1, \dots, p_M$  are positive integers dividing  $(2^M - 1)$ . Clearly, this phenomenon is novel and is never found in both the standard MG and HMG<sup>FSS</sup>. We observe the trend that long period dynamics tends to be more stable in the sense that it lasts longer. In fact, the longest periodic dynamics, namely the one with period  $2(2^M - 1)$ , appears to be the most stable. However, being the most stable dynamics does not necessarily mean that it must show up for every initial quenched disorder. Actually, the period  $2(2^M - 1)$  dynamics is harder and harder to find as  $M$  increases beyond about 7.

In the case of  $I \geq I^{c2}$ , the periodic dynamics of the orderly phase of the intermittency is weak compared with the case of  $I^{c1} \lesssim I < I^{c2}$ . As shown in Fig. 7c, the maximum period is in the form  $2^j(2^M - 1)$  where  $j$  is an integer greater than or equal to 2.

Our findings of the dynamics of the minority choice time series can be tabulated in Table I.

Period of dynamics	MG	HMG <sup>FSS</sup>		HMG <sup>MRSS</sup>	
		$I < I^{c1}$	$I > I^{c1}$	$I \ll I^{c1}$	$I \gtrsim I^{c1}$
$\alpha \gg \alpha_c^{\text{MG}}$	nil	nil	nil	nil	nil
$\alpha \approx \alpha_c^{\text{MG}}$	$2^{M+1}$	$2^{M+1}$	nil	$2^{M+1}$	intermittent <sup>a</sup>
$\alpha \ll \alpha_c^{\text{MG}}$	$2^{M+1}$	$2^{M+1}$	nil	$2^{M+1}$	nil <sup>b</sup>

<sup>a</sup>The maximum period of the orderly phase of this dynamics is  $2^j(2^M - 1)$  where the value of  $j$  can be found in the main text.

<sup>b</sup>But the minority choice time series conditioned on an arbitrary but fixed history shows a very weak period two dynamics

TABLE I: Summary of the dynamics in the minority choice time series for MG and HMG for odd number of agents and  $M \geq 2$ .

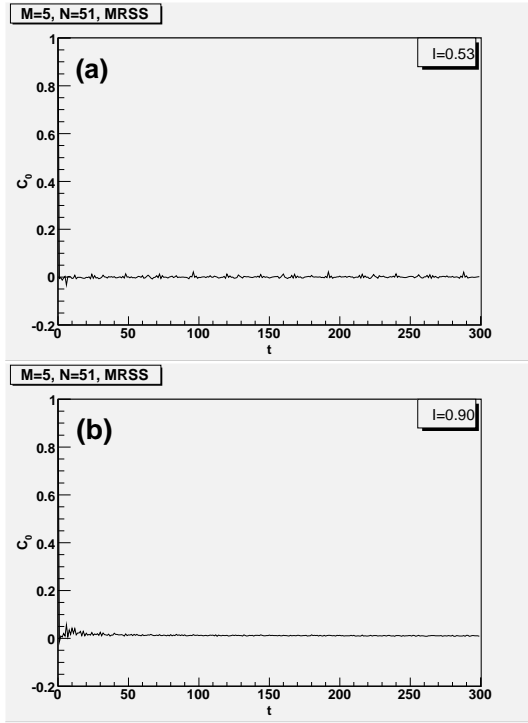


FIG. 4: The auto-correlation of HMG<sup>MRSS</sup> for  $\alpha = 0.63 \gg \alpha_c^{\text{MG}}$  when (a)  $I = 0.53$  and (b)  $I = 0.90$ .

#### D. Conditions for the period $2^j$ L.C.M. $(p_1, p_2, \dots, p_M)$ dynamics in the orderly phase of the HMG<sup>MRSS</sup> intermittency

We find that in HMG<sup>MRSS</sup>, some agents seldom change their strategies while others do so frequently. We say that an agent is **oscillating** if he switches strategy within the previous  $2^{M+1}$  turns. Otherwise, the agent is said to be **frozen**. It turns out that the number of frozen agents and their performance provide important information to allow us to understand the origin of the dynamics in the

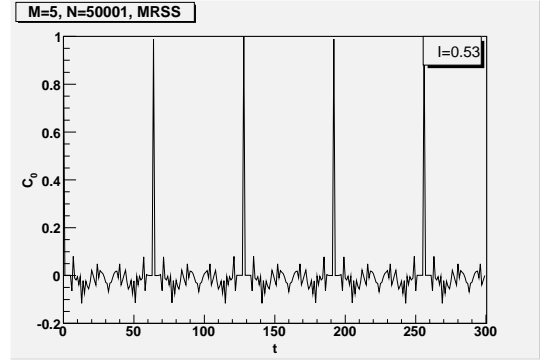


FIG. 5: The auto-correlation of HMG<sup>MRSS</sup> for  $I < I^{c1}$  and  $\alpha = 6.4 \times 10^{-4} \ll \alpha_c^{\text{MG}}$ .

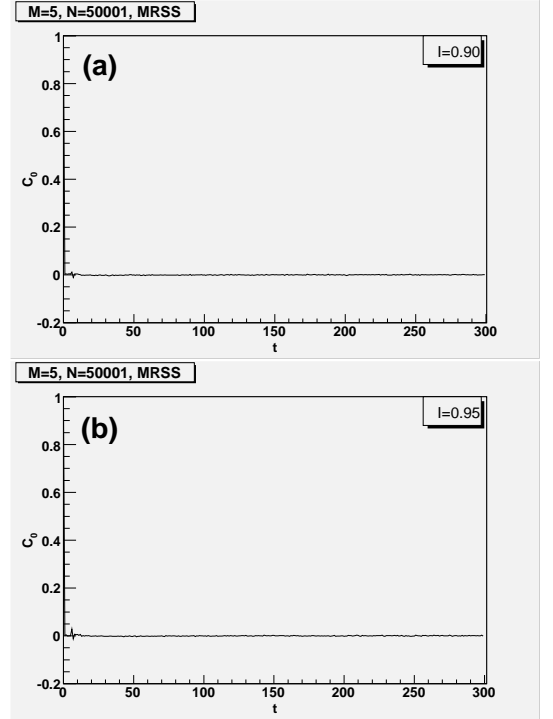


FIG. 6: The auto-correlation of HMG<sup>MRSS</sup> when  $\alpha = 6.4 \times 10^{-4} \ll \alpha_c^{\text{MG}}$  together with (a)  $I^{c1} < I < I^{c2}$  and (b)  $I \geq I^{c2}$ .

minority choice time series of HMG<sup>MRSS</sup> for  $I \gtrsim I^{c1}$  and  $\alpha \approx \alpha_c^{\text{MG}}$ .

Table II shows the average probabilities for a frozen (an oscillating) agent to correctly predict the minority side in a typical HMG<sup>MRSS</sup> with  $I \gtrsim I^{c1}$  and  $\alpha \approx \alpha_c^{\text{MG}}$ . Clearly, a frozen (an oscillating) agent will have a higher (lower) chance to correctly predict the minority side (that is, the winning probability) in the next turn. Since an oscillating agent must lose frequently in recent turns, our finding means that badly performing agents are likely to perform badly in future. More importantly, numerical simulations tell us that the presence of the period  $2^j$  L.C.M.  $(p_1, p_2, \dots, p_M)$  dynamics in the orderly phase of the intermittency where  $j \geq 1$  is an integer and

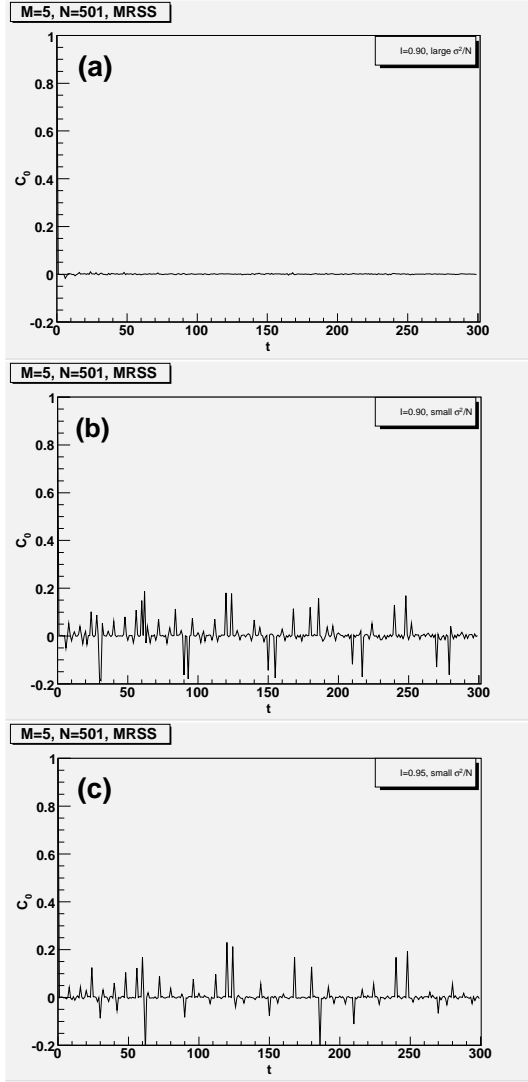


FIG. 7: The auto-correlation of  $\text{HMG}^{\text{MRSS}}$  for  $\alpha = 0.064 \approx \alpha_c^{\text{MG}}$  with (a)  $I^{c1} < I < I^{c2}$  and  $\sigma^2/N$  is large, (b)  $I^{c1} < I < I^{c2}$  and  $\sigma^2/N$  is small, and (c)  $I \geq I^{c2}$  and  $\sigma^2/N$  is small.

$N$	mean winning probability	
	frozen agents	oscillating agents
4001	0.499	0.292
8001	0.499	0.302
16001	0.499	0.293

TABLE II: The winning probability of frozen and oscillating agents for  $M = 3$  and  $I = 0.90$  averaged over 100 independent runs.

$p_1, \dots, p_M$  divide  $(2^M - 1)$  is almost always accompanied by the presence of only one oscillating agent in the entire system. Besides, this oscillating agent uses only two distinct strategies that are anti-correlated. (That is, these two strategies predict different minority side for all given histories [14, 15].) Moreover, the strategies used by the remaining frozen agents form a collection

of crowd-anticrowd pairs. Interestingly, our observed intermittency disappears and becomes a single aperiodic phase if we replace the histories by random variables or if the number of agents  $N$  is even. Also, the sole oscillating agent associated with each orderly phase of the time series may be different.

#### IV. THE CROWD-ANTICROWD EXPLANATION

Let us briefly review how the crowd-anticrowd theory explains the behavior of the standard MG before adapting it to explain the behavior of  $\text{HMG}^{\text{FSS}}$  and  $\text{HMG}^{\text{MRSS}}$ .

##### A. The crowd-anticrowd theory for the standard MG

According to the crowd-anticrowd theory, agent cooperation in the standard MG is determined by the number of (effective) anti-correlated pairs of strategies in current use. The smaller the difference between the number of agents currently adopting a strategy  $a$  and those currently adopting its anti-correlated strategy  $\bar{a}$ , the better the crowd-anticrowd cancellation leading to a better agent cooperation. Since the number of available strategies is much less than the strategy space size for  $\alpha \gg \alpha_c^{\text{MG}}$ , crowd-anticrowd cancellation cannot be effective in this regime. And because standard MG agents do not have inertia, they switch strategies immediately once the maximum virtual score difference is negative. Due to the fact that the virtual score of a strategy in the standard MG is independent of who owns or uses it, every standard MG agent has the same view on the performance of a given strategy. So, when  $\alpha \ll \alpha_c^{\text{MG}}$ , standard MG agents tend to adopt and drop similar strategies all the time. This over-reaction leads to a herd effect and is the origin of the maladaptation in the standard MG in this regime. Thus, effective agent cooperation is possible only for  $\alpha \approx \alpha_c^{\text{MG}}$  in the standard MG. A remarkable feature of the standard MG is that effective agent cooperation is indeed possible in this regime in spite of the fact that agents act independently by utilizing common global coarse-grained information only [14, 15].

##### B. Towards the crowd-anticrowd theory for HMG

Unlike the standard MG, HMG agents use Eq. (1) to decide whether to keep their currently using strategies or not. In general, they are initially assigned different strategies and begin to adopt their currently using strategies at different times. So, they have different virtual score difference  $\Delta\Omega_k$  and the number of turns since the adoption of the current strategy  $\tau_k$ . Together with the virtual score reset mechanism stated in rule 4 of HMG, the same strategy may be ranked differently

amongst HMG agents. Consequently, by picking strategies from the same strategy space, the effective strategy diversity for HMG is in general bigger than that for the standard MG. Furthermore, the higher the value of  $I$  (and hence the higher the inertia), the slower the average rate of strategy switching. All these factors reduce HMG agents' over-reaction and the herd effect making the system to better cooperate [11, 12]. Just like the standard MG, the dynamics of HMG encourages agents to form crowd-anticrowd pairs thereby increasing agent cooperation. But unlike the standard MG, this "crowd-anticrowd pair formation" driving force in HMG is more gentle and is less likely to cause over-reaction and maladaptation because of the presence of inertia. Hence, the higher the inertia, the longer the equilibration time. At this point, we have to emphasize that the presence of inertia need not imply that the agents must cooperate because the "crowd-anticrowd pair formation" driving force may not be strong enough in a certain parameter regime. The bottom line is that the presence of inertia never worsen the agent cooperation. In other words, for fixed values of  $M$ ,  $N$  and  $S$ , the variance of attendance per agent  $\sigma^2/N$  for the standard MG has to be greater than or equal to that for the  $\text{HMG}^{\text{FSS}}$  or  $\text{HMG}^{\text{MRSS}}$ .

Suppose the agents really cooperate. Still there are two ways to prevent them from cooperating forever. Consider a history  $\vec{\mu}$  which has non-zero probability of occurrence. Suppose further that the minority choice time series conditioned on this history  $\vec{\mu}$  is biased. (That is, it is more likely to find a particular minority choice than the other in this conditioned time series.) In this case, certain strategy  $a$  will outperform its anti-correlated partner  $\bar{a}$  in the long run. More precisely, the rate of change of virtual score difference between  $a$  and  $\bar{a}$  averaged over a sufficiently long number of turns is positive. So, after a sufficiently long time, agents will begin to drop strategy  $\bar{a}$  and adopt strategy  $a$ , making strategy  $\bar{a}$  much more popular than strategy  $a$ . In particular, if the minority choice time series exhibits periodic dynamics, then the timescale for a frozen agent to change to an oscillating agent via this mechanism is directly proportional to the period of the dynamics and inversely proportional to the number of biased histories in the minority choice time series.

Even if the minority choice time series conditioned on every history is un-biased, there is still a way for agents to stop cooperating. Provided that the value of  $\Delta\Omega_k$  follows an un-biased random walk, after sufficiently long time, agent  $k$  can switch his strategy once a while due to fluctuations in  $\Delta\Omega_k$ . These two mechanisms act like a "crowd-anticrowd pair destruction" driving force that decreases agent cooperation. Surely, the former mechanism is more efficient.

In summary, it is the combined actions of the above two driving forces that determine the agent cooperation and dynamics of HMG. In fact, the "crowd-anticrowd pair formation" driving force dominates the initial dynamics of the HMG. And the "crowd-anticrowd pair de-

struction" driving force becomes important after most of the agents have been paired up. This picture allows us to understand the simulation results reported in Sec. III.

### C. The crowd-anticrowd explanation for $\text{HMG}^{\text{MRSS}}$

In the case of  $\alpha \gg \alpha_c^{\text{MG}}$ , there are so few strategies at play that most of the crowd-anticrowd pairs is made up of only one agent. As a result, the "crowd-anticrowd pair formation" driving force is never strong enough to ensure agent cooperation irrespective of the value of  $I$ . Thus, crowd-anticrowd cancellation is ineffective. Besides, agents in effect make random choices each turn so that the value of  $\sigma^2/N$  approaches the coin-toss limit as  $\alpha \rightarrow \infty$ . Surely, the minority choice time series does not show any periodic dynamics [14, 15]. Since the above arguments are also valid for the standard MG and  $\text{HMG}^{\text{FSS}}$ , we understand why MG,  $\text{HMG}^{\text{FSS}}$  and  $\text{HMG}^{\text{MRSS}}$  all behave in the same way in this parameter regime.

In the case of  $I \lesssim I^{c_1}$ , the inertia of agents is so low that agents switch strategies immediately whenever the maximum virtual score difference is negative. In other words, the response of standard MG and HMG agents are the same in this parameter regime [11]. Hence, the agent cooperation and dynamics of standard MG,  $\text{HMG}^{\text{FSS}}$  and  $\text{HMG}^{\text{MRSS}}$  are about the same.

The remaining case to study is  $I \gtrsim I^{c_1}$  and  $\alpha \lesssim \alpha_c^{\text{MG}}$ . Note that the behavior of  $\text{HMG}^{\text{FSS}}$  and  $\text{HMG}^{\text{MRSS}}$  in this case differ markedly as reported in Sec. III. We begin our analysis by stating the following claim whose proof can be found in Appendix A.

**Claim 1** *Let  $N$  be an even number. Suppose each of the  $N$  players are randomly and independently assigned  $S$  socks; and a sock has  $2^M$  possible colors and can either be left or right. (Hence, there are  $2^{M+1}$  kinds of socks.) Suppose further that each kind of sock is selected with equal probability. Then, provided that  $2^{M+1} < NS$ , the probability that there exists a way to form  $N/2$  pairs of socks by picking exactly one sock from each of the  $N$  players is greater than or equal to  $1 - \beta/N$  for some positive  $\beta$  which is independent of  $N$ .*

Recall that two strategies are said to be anti-correlated if they always predict different minority side. And they are said to be uncorrelated if they have equal chance to predict the same minority side provided that each history occurs with equal probability. Thus, strategies in the MRSS consist of  $2^M$  pairs of anti-correlated strategies and that strategies picked from two distinct pairs are uncorrelated [14, 15].

By identifying the  $2^{M+1}$  different kinds of socks with the  $2^{M+1}$  different strategies in the MRSS, Claim 1 implies that for a sufficiently large  $N$  and for  $\alpha < 1$ , the probability of forming  $\lfloor N/2 \rfloor$  pairs of anti-correlated

strategies by picking one strategy from each agent's strategy pool is high. Surely, finding this solution requires communications amongst the agents. If the agents could keep on using his particular choice of strategy throughout the game, maximum agent cooperation would be attained and hence the theoretical minimum value of  $\sigma^2/N$  (that is  $\sigma^2/N = 0$  if  $N$  is even and  $1/4N$  if  $N$  is odd) would be resulted. In contrast, since the strategy space size of the FSS is exponentially larger than that of the MRSS in the large  $M$  limit, the condition  $\alpha < 1$  is not sufficient for agents to maximally cooperate in the above way should they pick strategies from the FSS. The correct condition in this case should be  $2^{2^M} < NS$  (and hence,  $\alpha \rightarrow 0^+$  in the thermodynamic limit).

Claim 1 only assures the existence of an optimal way of agent cooperation with a high probability. It remains to show that this maximum agent cooperation can be achieved with a high chance for  $\text{HMG}^{\text{MRSS}}$  under certain conditions. Recall from step 5 of the rules of HMG that agent  $k$  uses the maximum virtual score difference  $\Delta\Omega_k$  amongst all the strategies initially assigned to him to decide whether to change strategy or not. The following consequences of step 5 are important to understand the strong dependence of HMG on the strategy space used:

1. The higher the value of  $I_k$ , the more willingly for agent  $k$  to stick to his currently using strategy.
2. For fixed values of  $I_k$ 's, agents with a strategy and its anti-correlated partner in their pool of strategies have the strongest tendency, on average, to switch their strategies.
3. Suppose agent  $k$  has just switched to a new strategy and that this newly adopted strategy incorrectly predicts the minority side in its first use, then  $\tau_k = 1$  and  $\Delta\Omega_k = -2$ . As a result, this agent will drop his newly adopted strategy in the next turn provided that  $I_k < I^{c_2}$ .

1. The sub-case of  $I \gtrsim I^{c_1}$  and  $\alpha \approx \alpha_c^{\text{MG}}$

Shortly after the commencement of  $\text{HMG}^{\text{MRSS}}$ , the minority choice time series should resemble an un-biased random sequence. So provided that  $I$  is sufficiently large, Subsec. IV B tells us that the "crowd-anticrowd pair formation" driving force allows a large number of agents to form crowd-anticrowd pairs. Most of these paired agents will be frozen, and there are only a few oscillating agents in the system. By simple probability consideration, we expect that most of the strategies hold by these oscillating agents are un-correlated. As the dynamics of the system, which is determined mostly by the dynamics of these oscillating agents, the minority choice time series conditioned on an arbitrary history is likely to be un-biased. Hence, fewer and fewer oscillating agents will present as they gradually form crowd-anticrowd pair and become frozen. From Claim 1, we believe that for

a sufficiently large  $I$ , agents in  $\text{HMG}^{\text{MRSS}}$  have a high chance to attain maximum agent cooperation provided that  $\alpha \approx \alpha_c^{\text{MG}} < 1$ . This is consistent with the findings in our numerical simulations reported earlier in Sec. III that the highest chance of finding maximum agent cooperation is when  $\alpha \approx \alpha_c^{\text{MG}}$  and  $I \gtrsim I^{c_1}$ . And this maximum agent cooperation is accompanied by the existence of at most one oscillating agent in the system who switches between a pair of anti-correlated strategies. This finding agrees with the discussion following Claim 1 that the agent holding a pair of anti-correlated strategies switches his strategy most readily. To conclude, when  $N$  is odd, the effective number of strategies at play for  $\text{HMG}^{\text{MRSS}}$  in this regime is reduced to one in most of the time. Thus, the frozen agents have an average of  $1/2$  chance to correctly predict the minority side in the next turn while the only oscillating agent has no chance to do so. Consequently, the frozen agents are unlikely to switch their currently using strategies while the only oscillating agent is prone to strategy switching.

We now discuss the dynamics of the minority choice time series when  $N$  is odd. From the above discussions, it is clear that the system is in an orderly (chaotic) phase whenever it has one (more than one) oscillating agent. In addition, this oscillating agent is most likely to be switching between a pair of anti-correlated strategies. This oscillating agent always predicts the minority side incorrectly throughout the corresponding orderly periodic dynamics phase. Thus, from the discussions immediately after Claim 1, we conclude that this oscillating agent must drop his currently using strategy and switch to its anti-correlated counterpart in each turn provided that  $I^{c_1} \lesssim I < I^{c_2}$ . By identifying a strategy with a linear function of the historical minority choices  $\mu_i$ 's over the finite field  $GF(2)$  in the form of Eq. (3), we know that the difference between the linear functions associated with two anti-correlated strategies is equal to 1. So, whenever  $\alpha \approx \alpha_c^{\text{MG}}$  and  $I^{c_1} \lesssim I < I^{c_2}$ , the minority side in the  $n$ th turn  $\mu_n$  throughout this orderly periodic dynamics phase obeys

$$\mu_n = \sum_{i=1}^M \eta_i \mu_{n-i} + \eta_0 + n, \quad (6)$$

where  $\eta_i, \mu_i \in GF(2)$  and  $n$  denotes  $1 + 1 + \dots + 1$  ( $n$  terms).

We show in Appendix B the following theorem.

**Theorem 1** *The sequence  $\{\mu_n\}$  generated by the recursion relation in Eq. (6) is periodic. Its period is in the form  $2^j L.C.M.(p_1, p_2, \dots, p_M)$  where  $j \geq 1$ , and  $p_1, \dots, p_M$  are positive integers dividing  $(2^M - 1)$ . Moreover, the longest possible period for this sequence is  $2(2^M - 1)$  if  $M \geq 2$  and 4 if  $M = 1$ . Suppose the sequence is of maximum possible period and  $M \geq 2$ . Denote the history  $(\mu_{n-M+1}, \mu_{n-M+2}, \dots, \mu_{n-1}, \mu_n)$  by  $\vec{\mu}(n)$ . Then, the two histories of alternating 0's and 1's, namely,  $(0, 1, 0, 1, \dots)$  and  $(1, 0, 1, 0, \dots)$  appear in the sequence of history  $H = \{\vec{\mu}(n)\}_{n=1}^{2(2^M-1)}$  once; while all the*



other  $(2^M - 2)$  possible histories all appear in the history sequence  $H$  twice.

This theorem allows us to explain the period of the orderly phase of the time series in the case of  $I^{c_1} \lesssim I < I^{c_2}$  and  $\alpha \approx \alpha_c^{\text{MG}}$ . In particular, the proof tells us that the longest periodic dynamics in the orderly phase for  $M \geq 2$  is of period  $2(2^M - 1)$ . Nevertheless, periods satisfying Theorem 1 do not show up in the orderly phases equally frequently. If the number of turns between two consecutive occurrence of a history in an orderly phase is odd, then the minority time series conditioned on this history shows period two dynamics because the sole oscillating agent makes alternating prediction of the minority side each time when he is given the same history. In other words, the minority choice time series conditioned on this history is un-biased. In contrast, if the number of turns between two consecutive occurrence of a history in an orderly phase is even, the minority time series conditioned on this history is biased for it exhibits period one dynamics. Thus, the “crowd-anticrowd destruction” driving force discussed in Subsec. IV B will break the maximum agent cooperation in a time proportional to the period of this orderly phase and inversely proportional to the number of conditional minority choice time series that exhibits period one dynamics.

Theorem 1 tells us that for  $M \geq 2$ , the longest possible period of the orderly phase is  $2(2^M - 1)$ . This dynamics is present when the (degree  $M$ ) characteristic polynomial of the recursion relation (6), which is associated with the pair of anti-correlated strategies used by the sole oscillating agent, is primitive. In a single period, the two histories that consists of alternating 0's and 1's appear once while all other histories appear twice. Therefore, the number of turns between two consecutive appearance of each of the two histories consisting of alternating 0's and 1's equal the even number  $2(2^M - 1)$ . It is easy to show that the number of turns between two consecutive appearance of all other histories must be odd. (One way to do so is that if  $\vec{\mu}(k) = \vec{\mu}(k')$  with  $(k - k')$  being a positive even number, the homogeneous parts of the solutions of Eq. (6) for  $n = k - i$  and  $n = k' - i$  are equal whenever  $i = 0, 1, \dots, M - 1$ . And this is possible only when  $(2^M - 1) \mid (k - k')$ . As  $(k - k')$  is even, so  $k - k' \geq 2(2^M - 1)$ . Since all other histories occurs twice in a single period, the number of turns between two consecutive occurrence for them must be odd.) In conclusion, the  $2(2^M - 1)$  period dynamics can be found in the minority choice time series although it cannot be ever-lasting. Since out of the  $2^M$  possible minority choice time series conditional on a particular history, only two of them show period one rather than period two dynamics, the period  $2(2^M - 1)$  dynamics in the minority choice time series is quite stable in the sense that it lasts for a longer time. This also explains why the stability of this period increases with  $M$ .

As  $M$  increases while keeping  $S$  and  $\alpha$  fixed, the probability of having an agent in the system that holds a pair of anti-correlated strategy is about  $1 - [1 - S(S +$

$1)/2^{M+2}]^N \rightarrow 1 - e^{-(S+1)/2\alpha}$ . As  $\alpha \approx \alpha_c^{\text{MG}} \approx 0.3$  in the intermittent phase, it is not surprising that there is a high chance for the sole oscillating agent in this phase to switch between two anti-correlated strategies. Nonetheless, from the proof of Theorem 1, the corresponding probability of having an agent in the system that holds a pair of anti-correlated strategy that causes the period  $2(2^M - 1)$  dynamics is about  $1 - [1 - S(S + 1)\varphi(2^M - 1)/2^{2(M+1)}M]^N$ , where  $\varphi$  denotes the Euler-Totient function. By prime number theorem, this probability is equal to at least  $1 - [1 - S(S + 1)(2^M - 1)/2^{2(M+1)}M^{c+1}]^N \rightarrow 1 - e^{-(S+1)/2\alpha M^{c+1}}$  as  $M \rightarrow \infty$ , where  $c$  is a positive constant. This explains why as  $M$  increases, there is general trend that the period  $2(2^M - 1)$  dynamics orderly phase occurs less frequently.

From Theorem 1, the second longest period in the orderly phase is of period  $4(2^{M-1} - 1)$  for  $M \geq 2$ . This dynamics is associated with a characteristic polynomial in the form  $(\lambda - 1)p(x)$  where  $p(x)$  is a primitive polynomial of degree  $(M - 1)$ . Again, using the idea in the proof of Theorem 1, it is easy to check that the period  $4(2^{M-1} - 1)$  orderly phase is quite stable (although it is not as stable as the period  $2(2^M - 1)$  dynamics) as all but four conditional minority choice time series exhibit period one rather than period two dynamics.

For those orderly phases with shorter periods, the number of distinct histories present is small so that the effective diversity of the strategies at play are greatly reduced. Combined with the presence of, in general, a larger proportion of period one conditional minority choice time series in this dynamics, these shorter period dynamics are much less stable in the sense that they can last for a much shorter time.

The same analysis can be applied to the case when  $I \geq I^{c_2}$ . In this case, the only oscillating agent in the system switches his strategy once every few turns. Applying the same analysis as in the proof of Theorem 1, it can be shown that the maximum period of the orderly phase is  $2^j(2^M - 1)$  where  $j$  is the number of turns between the adoption and termination of a strategy for the sole oscillating agent. In this situation, the strength of a lot of period two dynamics in the conditional minority choice time series are weakened making the dynamics in the orderly phase less pronounced.

Note that one of the factors making the above intermittent behavior possible is that the number of agents  $N$  is odd. In case  $N$  is even, there are equal number of agents choosing each side. Hence, all agents are frozen and have 1/2 chance of correctly predicting the minority side. And the minority choice time series does not show any periodicity. This is exactly what we observe in our simulation.

Let us discuss more about the agent cooperation. Even for  $\alpha \approx \alpha_c^{\text{MG}}$  and  $I$  sufficiently large, the average value of  $\sigma^2/N$  is still a little bit higher than the theoretical minimum due to three reasons. First, a few initial quenched disorders does not allow maximum agent cooperation. Second, the system may be trapped in a non-maximally

cooperative state even though the initial quenched disorder allows maximum agent cooperation. Third, as we have pointed out in Subsec. IV B, the aperiodic phase of the intermittency is associated with the sub-maximal agent cooperation making the value of  $\sigma^2/N$  averaged over initial quenched disorder greater than the minimum value. So, it is not surprising to find that the values of  $\sigma^2/N$  for  $\text{HMG}^{\text{MRSS}}$  and  $\text{HMG}^{\text{FSS}}$  are about the same for  $\alpha \approx \alpha_c^{\text{MG}}$  and  $I \gtrsim I^{c1}$ .

## 2. The sub-case of $I \gtrsim I^{c1}$ and $\alpha \ll \alpha_c^{\text{MG}}$

For fixed  $M$  and  $S$ , the number of agents  $N$  increases as  $\alpha$  decreases. From the discussions in Subsec. IV B, we know for a sufficiently large  $N$ , the timescale for all except at most one of the agents to form crowd-anticrowd pairs is longer than the timescale for a pair of crowd-anticrowd agents to break up. Therefore, as  $\alpha \ll \alpha_c^{\text{MG}}$ , it is highly unlikely for the system to attain maximum agent cooperation. Besides, by further increasing  $\alpha$ , overcrowding of strategies is severe as more and more agents are using the same strategy with similar virtual scores and number of turns since its adoption. As a result, maladaptation and herd effect begin to appear. These are the reasons why for  $I \gtrsim I^{c1}$ , the value of  $\sigma^2/N$  starts to increase as  $\alpha$  falls below about 0.1. This is also the reason why the minority choice time series conditioned on an arbitrary but fixed history exhibits a very weak period two dynamics. However this period two dynamics is much weaker than the one observed in the standard MG because the presence of inertia makes the agents to response less readily. As the effective strategy diversity of  $\text{HMG}^{\text{FSS}}$  is greater than that of  $\text{HMG}^{\text{MRSS}}$  which is in turn greater than that of the standard MG, the value of  $\sigma^2/N$  for  $\text{HMG}^{\text{FSS}}$  is smaller than that of  $\text{HMG}^{\text{MRSS}}$  which is in turn smaller than that of the standard MG provided that  $I \gtrsim I^{c1}$ .

Discussions in the previous two sub-cases predict that by decreasing  $\alpha$  below  $\approx 2^{2^M}/NS$ , the value of  $\sigma^2/N$  for  $\text{HMG}^{\text{FSS}}$  will start to increase due to overcrowding of strategies. Unfortunately, we are not able to check the correctness of our prediction because the memory and run time requirements are too high.

## V. DISCUSSIONS

In summary, we have performed extensive numerical simulations to study the behavior of  $\text{HMG}^{\text{MRSS}}$ . We found that HMG agents cooperate better provided that their strategies are picked from the FSS instead of from the MRSS. This is because the effective diversity of strategies is higher in the former case. Based on the crowd-anticrowd theory [14, 15], we understood the origin of agent cooperation for  $\text{HMG}^{\text{MRSS}}$  in various parameter ranges by studying the interplay between the so-called “crowd-anticrowd pair formation” driving force

and the so-called “crowd-anticrowd pair destruction” driving force. And we found that the difference in the cooperative and dynamical behavior between  $\text{HMG}^{\text{FSS}}$  and  $\text{HMG}^{\text{MRSS}}$  is mainly due to the structure of the strategy space used. In particular, we were able to explain the novel intermittent behavior of the system and the orderly phase dynamics of the minority choice time series when  $\alpha \approx \alpha_c^{\text{MG}}$ ,  $I \gtrsim I^{c1}$ ,  $N$  is odd and  $\sigma^2/N \approx 1/4N$ . Essentially, this novel orderly dynamics in the intermittent phase is caused by the fact that the game is effectively reduced to a similar game played by only one agent most of the time and is accompanied by the maximum possible agent cooperation. And this reduction is possible in case the strategy pool is MRSS rather than FSS.

On one hand,  $\text{HMG}^{\text{MRSS}}$  appears to be special in the sense that it is the only variant of MG we have examined whose behavior depends sensitively on whether the FSS or the MRSS is used. On the other hand, the assumption that the behavior of standard MG when played in FSS or in MRSS is about the same is only a working assumption supported by numerical simulation results and heuristic arguments [14, 15]. Although this assumption greatly simplifies the space complexity of numerical simulations and sometimes even enable us to obtain a few semi-analytical results, one should bear in mind that this is only an assumption after all. In this regard, it is instructive to study the conditions under which one can replace FSS by MRSS without significantly affecting the dynamics and cooperative behavior of a system.

Lastly, we remark that the standard way of using the number  $2^{M+1}$  to measure the diversity of strategies is no longer appropriate for HMG. We believe that by suitably defining the diversity of strategies (and hence also the expression for  $\alpha$ ), the  $\sigma^2/N$  against  $\alpha$  curves for HMG played using different choices of strategy spaces can be made to coincide, at least roughly.

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## APPENDIX A: PROOF OF CLAIM 1

Let  $\Gamma, \Gamma'$  denote two different partitions of the  $N$  players into  $N/2$  pairs. Let  $\text{Pr}_\Gamma$  denotes the probability that it is possible for each of the  $N/2$  pairs of agents in the partition  $\Gamma$  to pick a sock of the same color but different side. Clearly,  $\text{Pr}_\Gamma$ 's are exchangeable. That is,  $\text{Pr}_\Gamma = \text{Pr}_{\Gamma'}$ . In

fact, for  $S \ll 2^{M+1}$ ,

$$\begin{aligned} \Pr_\Gamma &\approx \left\{ \frac{(2^{M+1})!}{2^{S(M+1)}(2^{M+1}-S)!} \left[ 1 - \left( 1 - \frac{S}{2^{M+1}} \right)^S \right] \right\}^{\frac{N}{2}} \\ &\approx \frac{S^N}{2^{MNS}}. \end{aligned} \quad (\text{A1})$$

By de Finetti theorem [20] and its generalization by Diaconis and Freedman [21], as long as  $0 < \Pr_\Gamma < 1$  for any finite  $N$ , the probability that there exists a way to form  $N/2$  pairs of socks by picking one sock from each player is lower bounded by  $\Pr(\infty) - \beta/N$  for some  $\beta > 0$  independent of  $N$ , where  $\Pr(\infty)$  denotes the same probability when  $N \rightarrow \infty$ .

So, to prove this claim, it suffices to show that  $\Pr(\infty) = 1$  whenever  $2^{M+1} < NS$  and  $S \geq 2$ . Under these two conditions, it is obvious that  $\Pr_\Gamma \in (0, 1)$  for any finite  $N$ . So, de Finetti theorem implies that  $\Pr_\Gamma$ 's are conditionally independent given the tail  $\sigma$ -field [20]. Thus,

$$\Pr(\infty) = \lim_{N \rightarrow \infty} 1 - \prod_\Gamma (1 - \Pr_\Gamma) = 1. \quad (\text{A2})$$

Hence, claim 1 is proved.  $\square$

Before leaving this appendix, we point out a common mistake people makes. The probability that all paired players have matching socks for an arbitrary but fixed partition  $\Gamma$  is equal to  $P(\Gamma) \approx \left[ 1 - (1 - S/2^{M+1})^S \right]^{N/2}$ . Thus, the expected number of partitions satisfying this ‘‘all paired players’’ condition is  $P(\Gamma) \times N! / [(N/2)!]^2$ . A threshold condition for the existence of a partition that all paired players have matching socks can then be deduced by saddle point approximation. The loophole in this argument is that socks are assigned once and for all to the players so that the probabilities  $P(\Gamma)$  and  $P(\Gamma')$  are not independent.

## APPENDIX B: PROOF OF THEOREM 1

This proof uses a number of finite field concepts and techniques. Readers who are not familiar may consult Ref. [22] before moving on. The characteristic equation of Eq. (6) is

$$\lambda^M - \sum_{i=1}^M \eta_i \lambda^{M-i} = 0. \quad (\text{B1})$$

If the characteristic equation has no degenerate root, the homogeneous part of the solution of  $\mu_n$  in Eq. (6) is in the form  $\sum_i g_i w_i^n$  where  $w_i$  are roots of Eq. (B1) over the extension field  $GF(2^M)$  for some  $g_i \in GF(2^M)$ . Clearly, the period of the sequence  $\{\sum_i g_i w_i^n\}_{n=0}^{+\infty}$  divides L.C.M.  $(|w_1|, |w_2|, \dots, |w_M|)$ , where  $|w_i|$  denotes the order of  $w_i$ . And the equality holds if  $g_i$ 's are all non-zero. (Remember also that  $|w_i|$  divides  $(2^M - 1)$  for all  $i$ .)

What if the roots of Eq. (B1) are degenerate? Suppose  $w_1$  is a double root of Eq. (B1). Then,  $w_1^n$  and  $nw_1^n$  are the two generators of the homogeneous part of the solution of Eq. (6). Besides, the period of the sequence  $\{g_1 w_1^n + g_2 n w_1^n\}$  divides  $2|w_1|$  with the equality holds if  $g_1, g_2 \neq 0$ . Similarly, let  $2^j$  be the smallest integer greater than or equal to  $k$ . Then, it is straight-forward to check that  $2^j|w_1|$  is divisible by the period of the homogeneous part of the solution of the recursion relation (6) that corresponds to a degree  $k$  root  $w_1$  of the characteristic equation (B1). Furthermore, the period can attain the value  $2^j|w_1|$  provided that the coefficients  $g_i$ 's are non-zero.

Combining the two cases above, we conclude that the period of the homogeneous solution of Eq. (6) divides  $2^j \text{L.C.M.}(|w_1|, |w_2|, \dots)$  where  $w_i$ 's are the distinct roots of its characteristic equation and  $j$  is a non-negative integer. The maximum possible period for this homogeneous solution is  $(2^M - 1)$ ; and this is attainable provided that the coefficient in each term of the homogeneous solution is non-zero and the characteristic polynomial of Eq. (6) is any one of the  $\varphi(2^M - 1)/M$  primitive polynomials of degree  $M$  in the  $GF(2)[\lambda]$ , where  $\varphi$  is the Euler-Totient function. Note further that if the minority choice time series were generated by the homogeneous solution of Eq. (6) alone, then this time series would not contain  $M$  consecutive 0's should its period be  $(2^M - 1)$ . Besides, in one period, all history strings but the one consists of all 0's would appear exactly once.

Now, we study the particular solution of Eq. (6). The first case to consider is when the characteristic polynomial has an odd number of (non-zero) terms. (This includes the case when the polynomial is irreducible and  $M \geq 2$  for otherwise 1 is a root of Eq. (B1).) In this case, a particular solution of Eq. (6) is  $\mu_n = \eta_0 + n \pmod{2}$  if  $|\{i : \eta_{n-2i} \neq 0\}|$  is even or  $\mu_n = \eta_0 + n + 1 \pmod{2}$  if  $|\{i : \eta_{n-2i} \neq 0\}|$  is odd. In either case, the period of this particular solution is 2.

The remaining case is when the number of (non-zero) terms in Eq. (B1) is even. In this case,  $(\lambda - 1)$  must be a factor of the R.H.S. of Eq. (B1). We write the R.H.S. of Eq. (B1) as  $(\lambda - 1)^k p(\lambda)$  for some polynomial  $p(\lambda)$  with  $p(1) \neq 0$ . If  $k = 1$ , a particular solution is  $\mu_n = \lfloor (n + 1 - \eta_0)/2 \rfloor \pmod{2}$  which is of period 4. Similarly, it is easy to check that there is a particular solution of Eq. (6) whose period divides  $2^j$  where  $2^j$  is the smallest integer greater than  $(k + 1)$ .

By combining the studies of the periods of the homogeneous and particular solutions of Eq. (6), we conclude that the period of the general solution of Eq. (6) is in the form  $2^j \text{L.C.M.}(|w_1|, |w_2|, \dots)$  where  $w_i$ 's are the distinct roots of Eq. (B1) and  $2^j$  is the smallest integer greater than  $(k + 1)$ . Moreover, the maximum possible period for the solution of the recursion relation (6) is  $2(2^M - 1)$  for  $M \geq 2$  and 4 for  $M = 1$ . In the former case, such a maximum period is attained only if the characteristic polynomial of Eq. (6) is primitive. And in the latter case, it is attained when the characteristic polynomial is  $\lambda - 1$ .

Finally, we consider the case when the period of the

sequence  $\{\mu_n\}$  attains its maximum possible value of  $2(2^M - 1)$ . In this case, a particular solution of Eq. (6) corresponds to the histories of alternating 0's and 1's. So, combined with our earlier analysis on the frequency of histories for the homogeneous part of the solution of

Eq. (6), we conclude that in each period, all but the two histories of alternating 0's and 1's appear twice. Besides, the two histories of alternating 0's and 1's appear once. This proves the theorem.  $\square$

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